1 Solution Method for the Radiation Hydrodynamcs with Gray-Diffusion approximation

The governing equations of radiation hydrodynamics with gray-diffusion approximation can be obtained in the first approximation in u/c. They express the near conservation of mass, momentum, total energy, and radiation energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{u}] = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u}\mathbf{u} + (p + p_{\mathrm{R}})\mathbf{I}] = 0, \tag{2}$$

$$\frac{\partial(\mathcal{E} + E_{\mathrm{R}})}{\partial t} + \nabla \cdot [(\mathcal{E} + E_{\mathrm{R}} + p + p_{\mathrm{R}})\, \mathbf{u}] = \nabla \cdot \left[\frac{c}{3\overline{\chi}} \nabla E_{\mathrm{R}}\right], \tag{3}$$

$$\frac{\partial E_{\mathrm{R}}}{\partial t} + \nabla \cdot [E_{\mathrm{R}} \mathbf{u}] + p_{\mathrm{R}} \nabla \cdot \mathbf{u} = \nabla \cdot \left[\frac{c}{3\overline{\chi}} \nabla E_{\mathrm{R}}\right] - \kappa_{\mathrm{P}} c (E_{\mathrm{R}} - \frac{4\sigma}{c} T^4), \tag{4}$$

where ρ, $\mathbf{u}$, p, T are the density, plasma velocity, gas kinetic pressure, and temperature, respectively. The total plasma energy density $\mathcal{E}$ is related to the internal plasma energy e:

$$\mathcal{E} = \frac{1}{2}\rho u^2 + \rho e. \tag{5}$$

The radiation field is assumed to be isotropic, so that the radiation pressure can be obtained from the radiation energy density E_{R}:

$$p_{\mathrm{R}} = \frac{1}{3} E_{\mathrm{R}}. \tag{6}$$

The radiation is treated as a fluid that carries momentum and energy. In essence, equations (1)–(4) describe a two temperature fluid. Two cross-sections have been introduced, namely the Planck mean opacity κ_{P} and an averaged opacity $\overline{\chi}$ that appears in the radiation diffusion coeffient. To close the dynamical equations we need equation of state data. If the radiation is negligible and the material is a polytropic gas, then the internal energy would be

$$\rho e = \frac{p}{\gamma - 1}, \tag{7}$$

where γ is fixed. In general, we can not fix the polytropic index due to radiation effects. In the following we will fix γ, but instead indicate the deviation in the internal energy from that of a polytropic hydro-gas by ρe:

$$\rho e = \frac{p}{\gamma - 1} + \Delta(\rho e). \tag{8}$$

This introduces however a new advection equation for this extra internal energy of the plasma

$$\frac{\partial}{\partial t}\Delta(\rho e) + \nabla \cdot [\Delta(\rho e)\mathbf{u}] = 0, \tag{9}$$

which we need to solve together with the other radiation hydrodynamics equations. The problem will be solved using the shock-capturing schemes of the BATSRUS code. The left hand side of equations (1)–(4) are in a form that resembles the pure hydro equations. We will fully exploit this feature, so that we can solve this with the hydro solvers of BATSRUS. For the analogy with the hydro equations with have to re-interpret in the hydro solver the pressure with the total gas kinetic and radiation pressure $p+p_{\mathrm{R}}$. The internal energy is now given by the total gas and radiation energy

$$\begin{aligned} \rho e &= \frac{p}{\gamma - 1} + \Delta(\rho e) + E_{\mathrm{R}}, \\ &= \frac{p + p_{\mathrm{R}}}{\gamma - 1} + \Delta(\rho e) + \frac{3}{2} p_{\mathrm{R}}, \end{aligned} \tag{10}$$

where we have fixed $\gamma = 5/3$. This indicates that we have to solve yet nother advection equation for $3p_{\mathrm{R}}/2 = E_{\mathrm{R}}/2$. This is however already accomplished by the left hand side of the radiation energy equation (4). The solution of our radiation hydrodynamics equations with gray diffusion approximation is obtained as follows:

- First solve the hyperbolic part of the equations, which now read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{u}] = 0, \tag{11}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot [\rho \mathbf{u}\mathbf{u} + (p + p_{\mathrm{R}})\mathbf{I}] = 0, \tag{12}$$

$$\frac{\partial}{\partial t}\Delta(\rho e) + \nabla \cdot [\Delta(\rho e)\mathbf{u}] = 0, \tag{13}$$

$$\frac{\partial E_{\mathrm{R}}}{\partial t} + \nabla \cdot [E_{\mathrm{R}}\mathbf{u}] + p_{\mathrm{R}}\nabla \cdot \mathbf{u} = 0, \tag{14}$$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2 + \frac{p + p_{\mathrm{R}}}{\gamma - 1}) + \nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \frac{p + p_{\mathrm{R}}}{\gamma - 1} + p + p_{\mathrm{R}}\right)\mathbf{u}\right] - \frac{1}{2}p_{\mathrm{R}}\nabla \cdot \mathbf{u} = 0, \tag{15}$$

where equations (11), (12), and (15) are solved by the hydrodynamic numerical scheme and the energy equations (13) and (14) are treated as advected scalar equations. The overall system contains two sound waves that are modified by the radiation pressure. Since we have chosen to fix γ on the maximum allowable value of 5/3, we can easily find an upper bound for the sound speed:

$$c^2 = \frac{\gamma(p + p_{\mathrm{R}})}{\rho}. \tag{16}$$

Using this wave speed for the numerical diffusion and determining the time step of the hyperbolic part of the equations helps to stabilize the scheme.

- The previous step provides an intermediate solution for the radiative energy density, denoted as E'_{R}. Using the deficit in the internal energy for the plasma and radiation energy density as found by equation (13) and (14), we can recover the true internal energy of the plasma (denoted as e'). By applying the equation of state for our materials of choice we obtain the updated plasma pressure and temperature.

- In the next stage we have to solve for the source terms in the energy equations. This amounts to solving a coupled system for the plasmas temperature and radiation temperature ($E_{\mathrm{R}} \propto T_{\mathrm{R}}^4$):

$$\rho\frac{\partial e(T)}{\partial t} = \kappa_{\mathrm{P}} c(E_{\mathrm{R}} - \frac{4\sigma}{c}T^4), \tag{17}$$

$$\frac{\partial E_{\mathrm{R}}}{\partial t} = \nabla \cdot \left[\frac{c}{3\overline{\chi}}\nabla E_{\mathrm{R}}\right] - \kappa_{\mathrm{P}} c(E_{\mathrm{R}} - \frac{4\sigma}{c}T^4) \tag{18}$$

and advance solution through time step with initial conditions e' for the plasma internal energy and E'_{R} for the radiation energy.

2 Semi-Implicit Scheme

We describe here the scheme to solve equations (17) and (18) implicitly. Discretizing in time leads to

$$
\begin{aligned}
E_{\rm I}^{n+1} &= E_{\rm I}^{*} + \Delta t K^{*}\left[E_{\rm R}^{n+1} - (aT^4)^{n+1}\right] && (19)\\
E_{\rm R}^{n+1} &= E_{\rm R}^{*} - \Delta t K^{*}\left[E_{\rm R}^{n+1} - (aT^4)^{n+1}\right] + \Delta t \nabla\cdot\left[D^{*}\nabla E_{\rm R}^{n+1}\right] && (20)
\end{aligned}
$$

where time level $*$ corresponds to the state after the hydro update, and we introduced the following notation for internal energy, energy coupling, radiation diffusion and radiation energy coefficients: $E_{\rm I} = \rho e$, $K = \kappa_{\rm P} c$, $D = \frac{c}{3\chi}$, and $a = 4\sigma c$. Notice that the coupling and diffusion coefficients are taken at time level $*$ (frozen coefficients). This leads to a temporally first order scheme in general (unless the coefficients are constants in time). One can either

1. solve the coupled system of equations (19) and (20) implicitly or
2. solve equation (19) for $E_{\rm I}^{n+1}$, substitute the solution back into (20), and solve the resulting scalar equation (20) implicitly.

Both approaches involve the linearization of the $E_{\rm I}(T)$ function.

Here we describe the second scheme, because it is more efficient. Note that if we had heat conduction in (19), then we would have to solve the coupled system of equations.

First we introduce the Planck function $B = aT^4$ as a new variable, and replace $E_{\rm I}$ with it using the chain rule

$$
\frac{\partial E_{\rm I}}{\partial t} = \frac{\partial E_{\rm I}}{\partial T}\frac{\partial T}{\partial B}\frac{\partial B}{\partial t} = \frac{c_V}{4aT^3}\frac{\partial B}{\partial t} \tag{21}
$$

Now equation (19) can be replaced with

$$
B^{n+1} = B^{*} + \Delta t K'\left[E_{\rm R}^{n+1} - B^{n+1}\right] \tag{22}
$$

where

$$
K' = K^{*}\frac{4aT^3}{c_V} \tag{23}
$$

Equation (22) can be solved for

$$
B^{n+1} = \frac{B^{*} + \Delta t K' E_{\rm R}^{n+1}}{1 + \Delta t K'} \tag{24}
$$

This can be substituted into equation (20) to obtain

$$E_{\mathrm{R}}^{n+1} = E_{\mathrm{R}}^{*} - \Delta t K^{*} \left[E_{\mathrm{R}}^{n+1} - \frac{B^{*} + \Delta t K' E_{\mathrm{R}}^{n+1}}{1 + \Delta t K'} \right] + \Delta t \nabla \cdot \left[D^{*} \nabla E_{\mathrm{R}}^{n+1} \right] \quad (25)$$

and simplified to

$$E_{\mathrm{R}}^{n+1} = E_{\mathrm{R}}^{*} - \Delta t K'' \left[E_{\mathrm{R}}^{n+1} - B^{*} \right] + \Delta t \nabla \cdot \left[D^{*} \nabla E_{\mathrm{R}}^{n+1} \right] \quad (26)$$

where

$$K'' = \frac{K^{*}}{1 + \Delta t K'} \quad (27)$$

It is convenient to introduce $\Delta E_{\mathrm{R}} = E_{\mathrm{R}}^{n+1} - E_{\mathrm{R}}^{*}$ and rearrange equation (26) as

$$\left[\frac{1}{\Delta t} + K'' - \nabla \cdot D^{*} \nabla \right] \Delta E_{\mathrm{R}} = -K'' \left[E_{\mathrm{R}}^{*} - B^{*} \right] + \nabla \cdot \left[D^{*} \nabla E_{\mathrm{R}}^{*} \right] \quad (28)$$

We solve equation (28) for ΔE_{R} using a linear solver, update $E_{\mathrm{R}}^{n+1} = E_{\mathrm{R}}^{*} + \Delta E_{\mathrm{R}}$ and then either

1. obtain B^{n+1} from (24), and then get T^{n+1} and E_{I}^{n+1}, or
2. use a conservative update for the internal energy

The second choice is

$$E_{\mathrm{I}}^{n+1} = E_{\mathrm{I}}^{n} + \Delta t K'' \left[E_{\mathrm{R}}^{n+1} - B^{*} \right] \quad (29)$$

which conserves the total energy to round-off error.

2.1 Crank-Nicolson scheme

We can attempt to go second order in time with the assumption of (temporally) constant coefficients using the Crank-Nicolson scheme with $\beta = 1/2$:

$$
\begin{aligned}
B^{n+1} &= B^* + \beta\Delta t K' \left[E_{\mathrm{R}}^{n+1} - B^{n+1}\right] + (1-\beta)\Delta t K' \left[E_{\mathrm{R}}^* - B^*\right] \quad (30)\\
E_{\mathrm{R}}^{n+1} &= E_{\mathrm{R}}^* - \beta\Delta t K^* \left[E_{\mathrm{R}}^{n+1} - B^{n+1}\right] - (1-\beta)\Delta t K^* \left[E_{\mathrm{R}}^* - B^*\right]\\
&\quad + \beta\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^{n+1}\right] + (1-\beta)\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^*\right] \quad (31)
\end{aligned}
$$

We can solve (30) for

$$
B^{n+1} = \frac{B^* + \beta\Delta t K' E_{\mathrm{R}}^{n+1} + (1-\beta)\Delta t K' (E_{\mathrm{R}}^* - B^*)}{1 + \beta\Delta t K'} \quad (32)
$$

This can be substituted into equation (31) to obtain

$$
\begin{aligned}
E_{\mathrm{R}}^{n+1} &= E_{\mathrm{R}}^* - \beta\Delta t K^* \left[E_{\mathrm{R}}^{n+1} - \frac{B^* + \beta\Delta t K' E_{\mathrm{R}}^{n+1} + (1-\beta)\Delta t K' (E_{\mathrm{R}}^* - B^*)}{1 + \beta\Delta t K'}\right]\\
&\quad - (1-\beta)\Delta t K^* \left[E_{\mathrm{R}}^* - B^*\right]\\
&\quad + \beta\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^{n+1}\right] + (1-\beta)\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^*\right] \quad (33)
\end{aligned}
$$

and can be simplified to

$$
\begin{aligned}
E_{\mathrm{R}}^{n+1} &= E_{\mathrm{R}}^* - \beta\Delta t K'' \left[E_{\mathrm{R}}^{n+1} - B^*\right] - (1-\beta)\Delta t K'' \left[E_{\mathrm{R}}^* - B^*\right]\\
&\quad + \beta\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^{n+1}\right] + (1-\beta)\Delta t \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^*\right] \quad (34)
\end{aligned}
$$

where

$$
K'' = \frac{K^*}{1 + \beta\Delta t K'} \quad (35)
$$

Equation (34) can be rearranged to

$$
\left[\frac{1}{\Delta t} + \beta K'' - \beta\nabla \cdot D^* \nabla\right] \Delta E_{\mathrm{R}} = -K'' \left[E_{\mathrm{R}}^* - B^*\right] + \nabla \cdot \left[D^* \nabla E_{\mathrm{R}}^*\right] \quad (36)
$$

Note that the only difference relative to the backward Euler scheme is the presence of β in equations (36) and (35).

Equation (36) is solved for ΔE_{R}, and we update $E_{\mathrm{R}}^{n+1} = E_{\mathrm{R}}^* + \Delta E_{\mathrm{R}}$, one can either calculate B^{n+1} from equation (32), or do a conservative update for the internal energy

$$
E_{\mathrm{I}}^{n+1} = E_{\mathrm{I}}^n + \beta\Delta t K'' \left[E_{\mathrm{R}}^{n+1} - B^*\right] + (1-\beta)\Delta t K'' \left[E_{\mathrm{R}}^* - B^*\right] \quad (37)
$$

3 Boundary conditions

The radiation energy is strongly diffusive therefore the boundary conditions determine the solution more than the initial conditions. We apply the zero incoming flux condition satisfying

$$E_r + \frac{2D}{c}\mathbf{n} \cdot \nabla E_r = 0 \qquad (38)$$

where E_r is the radiation energy density, c the speed of light, $D = c/(3\kappa_R)$ is the diffusion coefficient based on the Rosseland mean opacity κ_R, and $\mathbf{n}$ is the outward pointing normal unit vector. For the left boundary this can be disretized as

$$\frac{E_0 + E_1}{2} - \frac{2D}{c}\frac{E_1 - E_0}{\Delta x} = 0 \qquad (39)$$

where we dropped the subscript r and replaced it with the cell index. Index 1 corresponds to the last physical cell and 0 to the ghost cell. This equation can be solved for the ghost cell value

$$E_0 = \frac{2D/(c\Delta x) - 1/2}{2D/(c\Delta x) + 1/2}E_1 \qquad (40)$$

For very small opacity D becomes very large, so the ratio will approach one, i.e. we get a zero gradient condition. If we take into account the flux limiter, the maximum value for D is

$$D = cE_r/|\nabla E_r| \qquad (41)$$

If we substitute this into the first equation of this section, we get

$$E_r - 2E_r = 0 \qquad (42)$$

assuming that $\mathbf{n} \cdot \nabla E_r$ is negative, i.e. the radiation energy is decreasing outside the boundaries. The factor 2 is probably not correct in this limit. In any case this does not provide a useful boundary condition. This means that in the free streaming limit other assumptions are required.